On the Connection between Statical and Dynamical Chaos

M. S. El Naschie and S. Al Athel

Z. Naturforsch. **44a**, 645–650 (1989); received March 18, 1989

The paper discusses the interrelationship between statical chaos and dynamical initial value problems. It is pointed out that approximate homoclinic and heteroclinic solitons can be perturbed to produce spacial asymptotic chaos in some buckled structural elastic systems which constitute strictly speaking boundary value problems.

Introduction

In marked contrast to initial value problems of dynamics, boundary value problems seem to preclude the possibility of chaos. Superficially the argument seems to be as compelling as it is simple. For chaos to take place, infinite time is needed [1]. Boundary value problems are by definition finite and consequently elastostatical, and structural systems cannot display chaos. Nevertheless experience teaches us that quasistatical systems with finite dimensions can sometimes display great complexity [2]. To understand this apparent contradiction, we need only to extend the notion of time to that of time-like co-ordinates. The well known statical-dynamical analogy of Kirschoff provides the frame work for such an extension [3, 4]. In what follows we will attempt to show that by virtue of this analogy and the use of a kind of stretched spacial co-ordinate, boundary value problems could exhibit a considerable complexity and tend asymptotically toward quasistatical chaotic states [5–8].

1. The Forced Pendulum – An Initial Value Problem

1.1. Parametric Excitation

The equation of motion of a mathematical pendulum which is forced periodically in the vertical direction at the pivot is easily found from the equilibrium condition to be

$$\overset{**}{\phi} - (\Omega^2/\omega^2) \sin \phi = (a/l) \sin \tau \sin \varphi ,$$

where ϕ is the angle of rotation, l the length of the pendulum, ω the frequency of excitation, $\Omega = \sqrt{g/l}$ the natural frequency, g the earth acceleration, a the amplitude of excitation, $\tau = \omega t$ the non-dimensional time and $(*) = \mathrm{d}(\,)/\mathrm{d}\tau$. Since the corresponding homogeneous differential equation describes a homoclinic orbit [7, 8] for the initial conditions $\phi = \pi$ and $\overset{*}{\phi} = 0$, then following Poincaré's homoclinic criterion one expects the forcing to initiate a chaotic motion near the hyperbolic saddle, i.e. the inverted dynamical unstable position. The next step is to use the most direct and simplest characterization of possible chaos, namely the Poincaré map method. Since we are dealing with an initial value problem, all what is needed is a simple numerical integration of the equations using a marching technique. Some of the Poincaré maps obtained for system parameters

$$\Omega^2/\omega^2 = 1; \qquad a/l = 0.01 \quad \text{and}$$
$$\Omega^2/\omega^2 = 0.0272222; \quad a/l = 0.15 ,$$

and various initial conditions are shown in Figure 1. They are quite comprehensive and show, may be for the first time for this very simple conservative problem, the behaviour anticipated by the celebrated KAM theorem [7, 8].

1.2. Simple Periodic-moment Excitation

In anticipation of the use of the statical-dynamical analogy in studying the elastica, it is essential to consider once more the preceding problem. However, this time the system will be excited externally by a simple periodic moment, so that the corresponding differential equation would simplify to

$$\overset{**}{\phi} - (\Omega^2/\omega^2) \sin \phi = a \sin \omega t .$$

Again following Poincarés homoclinic criterion, the system may be shown to be susceptible to chaotic motion, which is easily shown to be the case using the Poincaré map method in conjunction with an appropriate numerical technique. The relevant Poincaré

Reprint requests to Prof. Dr. M. S. El Naschie, 51 Pitt Place, Church Street, Epsom, Surrey KT17 4PY, England, U.K.

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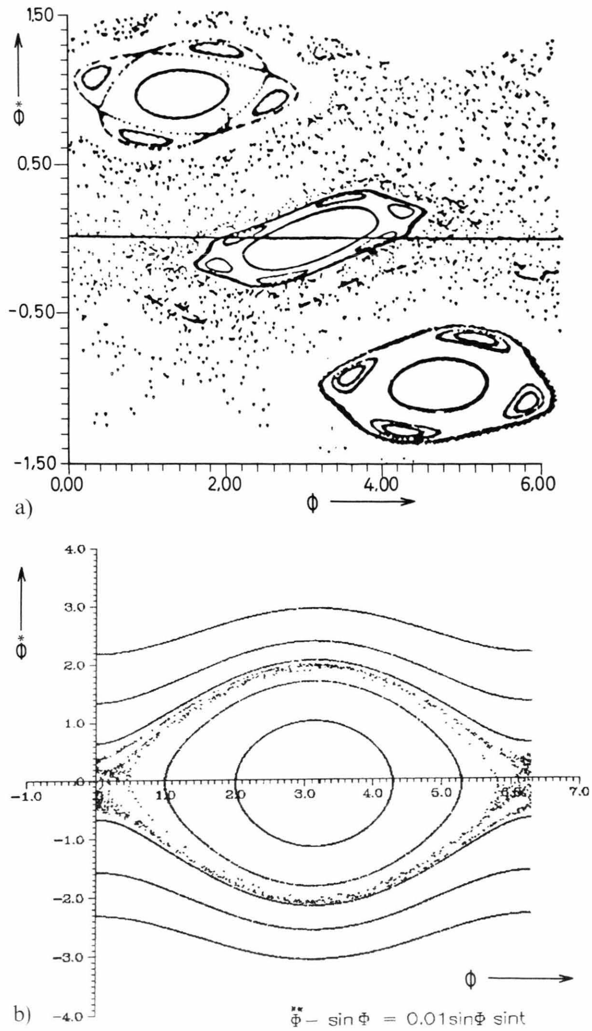

Fig. 1. Poincaré maps with various initial conditions for the parametrically excited pendulum in case of
(a) $\Omega_2^2/\omega_2^2 = 0.0272222$ and $a/l = 0.15$,
(b) $\Omega^2/\omega = 1$ and $a/l = 0.01$.

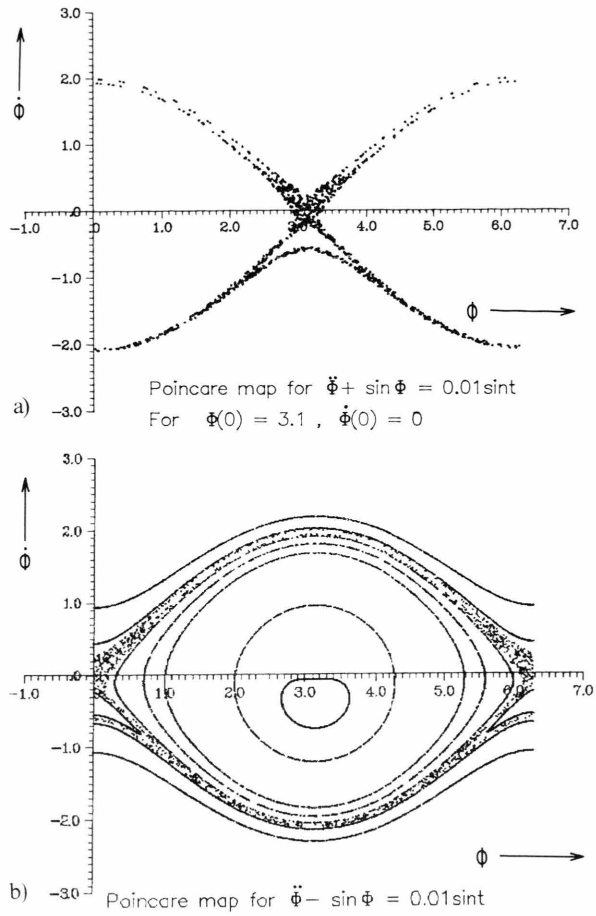

Fig. 2. Poincaré maps for the pendulum with periodic moment excitation and system values $\Omega^2/\omega^2 = 1$ and $a/l = 0.01$ in case of
(a) $\phi(0) = 3.1$ and $\dot{\phi}(0) = 0$,
(b) fifteen different initial values.

maps are shown in Figure 2. We may note at this stage that the shape of the maps was found to be extremely sensitive towards the step size of numerical integration and great care must be exercised to avoid wrong conclusions.

2. The Elastica – A Boundary Value Problem

Next we consider the differential equation of elastica [9]

$$\phi'' - \lambda^2 \sin\phi = 0$$

in terms of the angle of rotation ϕ, where $\lambda = \sqrt{P/\alpha}$, P is the axial load, α the bending stiffness, $(') = \mathrm{d}(\,)/\mathrm{d}s$, and s is the arch length. We see clearly that in this form the differential equation is mathematically identical to that of the unforced pendulum and that s plays the same role as the time in dynamics. Subsequently we suppose that the strut middleaxis possesses an initial sinusoidal crookedness. Taking only the first order effect of this shape imperfection, one finds

$$\alpha(\phi + a \sin\omega s)'' - P \sin\phi = 0 .$$

That means

$$\phi'' - \lambda^2 \sin\phi = a\,\omega^2 \sin\omega s .$$

Differentiating with respect to the time-like spacial co-ordinate ωs, where ω is the frequency of axial im-

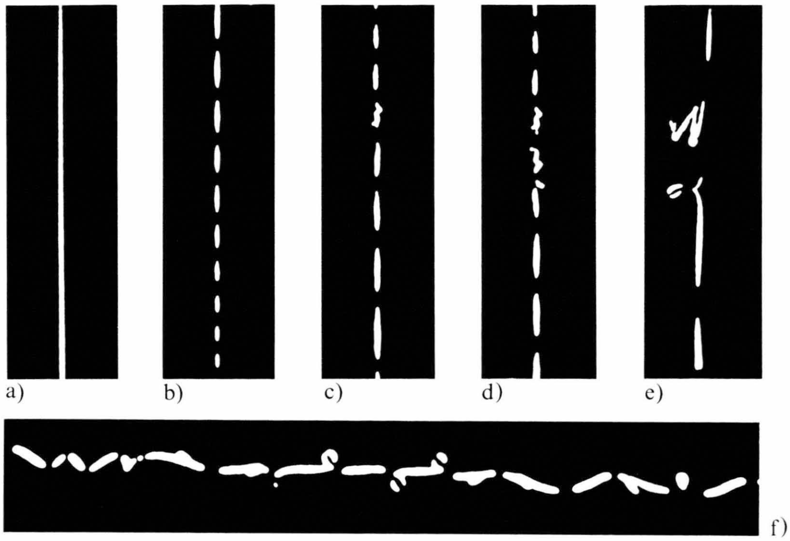

Fig. 3. Initiation of soliton-like asymptotic chaos in very long metal steel band with 3 D spacial imperfection. Note that in these simple physical experiments homoclinic as well as heteroclinic points are now possible. Exact mathematical conditions for the appearance of each one of these points may be established. – (a) The perfect band. – (b) The imperfect band. – (c) The first appearance of loops. – (d), (e), (f) Subsequent quasi random soliton loops. Note the similarity to the soliton found in plasma by Ichikawa et al. [21].

perfection, one finds

$$\ddot{\phi} - (\lambda^2/\omega^2)\sin\phi = a \sin \omega s,$$

where $(\dot{\ }) = \mathrm{d}(\ )/\mathrm{d}\omega s$ and a is the amplitude of axial shape imperfection. Now, that we do not need to consider more than the first order imperfection is theoretically founded on Koiter's theory of initial post-buckling [9–11] and in a wider mathematical context on catastrophe theory [11]. It is then important to realize that the differential equation of sinusoidally imperfect elastica, and the pendulum which is excited externally by a periodic moment, are mathematically identical. The only difference is that the first is a boundary value problem whilst the second is an initial value problem.

3. Chaos in Initial and Boundary Value Problems

A boundary value problem has two major consequences from the present discussion point of view. The first is regarding its finiteness, which precludes chaos by definition, as previously indicated. Second it cannot be integrated numerically using a direct marching technique. However, the first point could be bypassed by the physical assumption of infinitely long elastica, so that chaos cannot be excluded a priori. The second point is subsequently resolved automatically since the assumption of an infinite spacial domain and the analogy to the dynamical pendulum problem allows us to use a direct marching technique, exactly as in the case of an initial value problem. Consequently, the dynamical chaos of the moment excited pendulum near to the inverted position can be reinterpreted now in the spacial domain as a random formation of strophoid-like loops which were described for the first time by Leonard Euler in the appendix to his famous treatise on the calculus of variations (see Fig. 10, Tabular IV of [12]). These loops in turn can be interpreted within two extremely important fields of current mathematical research. First they may be viewed as a spacial-statical homoclinic connection formed by a rod of infinite length undergoing exceedingly large deformation in the limiting case when the angle tends towards $\phi_0 = \pi$ and a single loop is formed in the rod, a situation which corresponds in the dynamical analogy to the pendulum starting close to the inverted position of unstable equilibrium and making just one revolution as explained in great detail in the classical treatise of Love [3]. The second interpretation of these loops is that of so called solitons, which are in general localized wave solutions of permanent form [13, 14]. It is spacial imperfection which perturbs these solitary homoclinic loops into chaos so that they may appear at random in the theoretically infinitely long elastica (see Figs. 4 and 5). Nevertheless, long as elastica may be they are of course of finite length and consequently we

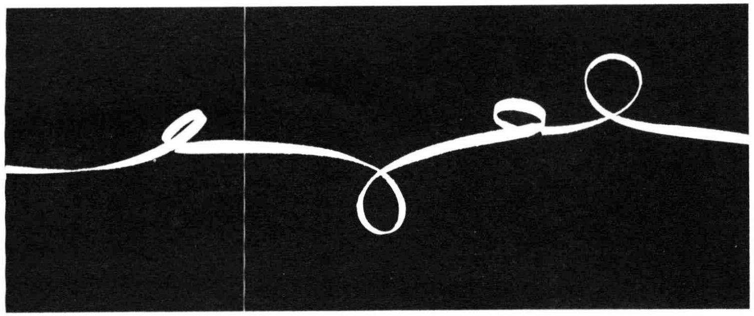

Fig. 4a. A photograph of some random soliton loops in a compressed very long elastic metal tape with periodic torsional imperfection.

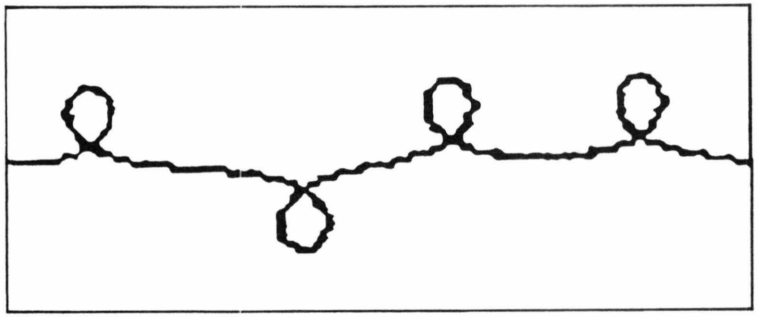

Fig. 4b. Result of numerical integration of the periodically imperfect planar elastica ($\ddot{\phi} - \sin\phi = 0.01 \sin t$; $\phi(0) = 6.2$; $\dot{\phi}(0) = 0.2$).

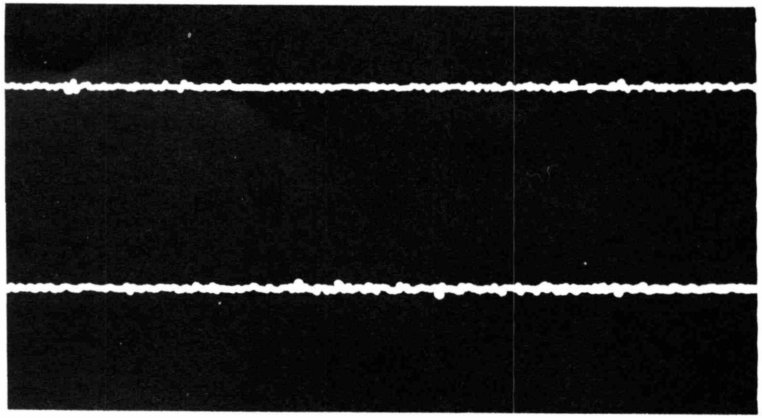

Fig. 5a. A photograph of spatial random knots in a long stretched and twirled rubber band as the stretching force is gradually released.

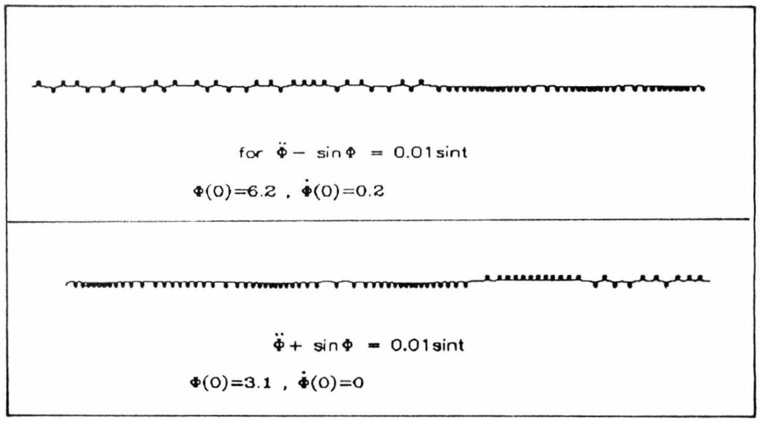

Fig. 5b. Result of numerical integration of the infinitely long periodically imperfect elastica.

may anticipate in general only a complex behaviour which is tending in theory towards a limiting chaos, but never truly reaches it. The consequence of such behaviour seems to have been foreseen for the first time by O. E. Roessler and his associates in a completely different context [5]. Such chaos may thus be termed asymptotic chaos. It is noteworthy, however, that we do not need the exceedingly large deflection in order to obtain the solitary loops if we remove the constraint of two dimensions and admit a third one. This fact may be readily demonstrated by a very simple experiment using a long fibreoptic string or a steel tape measure. Giving the tape a spacial imperfection, a small compression is sufficient to produce several such approximate homoclinic or heteroclinic points at random in the tape. In fact, the statical dynamical analogy of Kirchoff was extended long ago in 3 D by Hess [3], and the existence of spacial homoclinic and heteroclinic points [15] can be established. These are in turn the source of asymptotic chaos in 3 D elastica. Figure 3 shows the experimentally observed progressively complex appearance of an approximate homoclinic soliton in a long steel band*. This way many dynamical chaos problems [16–20] may be reinterpreted spacially. In Figs. 4 and 5 comparison is made between numerical integration and simple illustrative experiments of spacial asymptotic chaos.

* Heteroclinic points are now easily formed when the third dimension of the real physical space is admitted. Our model is thus a mechanical realization of soliton and chaos simultaneously.

4. Localized Damped Buckling Forms as Solitons

In conclusion it is important to remark that localized damped buckling forms also represent another form of a solitary homoclinic situation of the spacial-statical type which may exist in a boundary value problem and may also be perturbed by spacial imperfection into chaos [6–8]. The solitary solution acts thus in a sense as a sort of stretched co-ordinate, necessary to accommodate the infinity which chaos requires. The connection to buckling of shell-like structures is a more or less obvious application of this conjecture. A modified version of the inverse scattering method can be used to prove the existence of localized envelope soliton solutions for shell-like structures, but this will be given elsewhere.

5. Conclusion

The statical counterparts of Poincaré's homoclinic orbits can be perturbed by shape imperfection to produce a random sequence of localized soliton-like deformations. It is shown that boundary value problems can display asymptotic spacial chaos in the sense of Roessler, which is relevant to a deep understanding of localized buckling of elastic and shell-like structures.

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